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Hamiltonian Systems in Dynamic Reconstruction Problems ⋆

Nina N. Subbotina *

* *N.N. Krasovskii Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences (IMM UrB RAS) 16 S.Kovalevskaya str., Yekaterinburg, 620990 Russia; Ural Federal University named after the first President of Russia B.N.Yeltsin, 620002, 19 Mira str, Yekaterinburg, Russia; (e-mail: subb@uran.ru)*

Abstract: We consider dynamic reconstruction (DR) problems for controlled dynamical systems linear in controls and nonlinear in state variables as inaccurate current information about real motions is known. A solution of this on-line inverse problem is obtained with the help of auxiliary problems of calculus of variations (CV) for integral discrepancy functionals. Key elements of the constructions are solutions of hamiltonian systems obtained with the help of optimality conditions for the CV problems.
An illustrating example is exposed.

Keywords: dynamic reconstruction, calculus of variations, hamiltonian systems

1. INTRODUCTION

When analyzing models of controlled processes in economy, engineering, biology, medicine and so on, there often arises the need to solve reconstruction problems for real controls and trajectories basing on inaccurate measurements of motions of these models. A large number of methods for solving these inverse problems are developed for the static case, when information about the measurements of motions comes a posteriori, after the control process is completed. (See, for example, Leitmann (1962); Letov (1969); Michel (1977); D'Autilia1 and Bozzini (2017)).

Further, with the expansion and complication of applied problems, the inverse problems of dynamic reconstruction (DR) of controls and trajectories come to the fore. It is required to extract solutions of these problems on-line basing on current information about inaccurate measurements of motions of the models.

In the paper, controlled dynamical systems are considered which are linear in controls and nonlinear in state variables. Current information about measurements of real states of the system are inaccurate, with known error estimates. The problem of on-line reconstruction of real controls is solved.

The most close to the method presenting in this paper is the well-known approach for solving inverse problems that was proposed in studies by Osipov and Kryazhimski Kryazhimskii and Osipov (1984); Osipov and Kryazhimskii (1995), (see, also, Osipov and Maksimov). The approach involves a regularized procedure of extremal aiming at the dynamics of a guiding system similar to the considered basic dynamical system. So, the construction uses the couple system of double state variables. This approach appels to the optimal feedback theory developed in N.N. Krasovskii school Krasovskii (1968); Krasovskii and Subbotin (1988).

In the paper a new method Subbotina (2013) is introduced and discussed. In contrast with the first approach, auxiliary problems of calculus of variations (CV) for an integral regularized discrepancy functional are considered. Optimality conditions of CV in terms of hamiltonian system for state and conjugate variables are applied to solve DR problem. So, the suggesting constructions use the coupled system of state and conjugate variables.

Note that both above mentioned approaches solving inverse problems for controlled systems can be regarded as variants of Tikhonov's regularization method Tikhonov (1943).

Firstly, the new method was applied to solve inverse problems formulated in a static setting (reconstruction of controls and identification of parameters), where a posteriori information is available. (See, Subbotina (2014, 2015); Subbotina et al. (2015); Subbotina and Krupennikov (2016); Subbotina et al. (2016); Subbotina and Krupennikov (2017)).

A distinctive feature of applications of the method is that the negative discrepancies Subbotina (2017); Subbotina and Krupennikov (2017) were implemented. We have constructed coupled hamiltonian systems more stable to disturbances of input data than hamiltonian systems obtained as a result of applications of positive discrepancies.

In this paper, a new algorithm for solving problems of dynamic reconstruction is presented. To solve the problem of dynamic reconstruction of real controls, we introduce

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auxiliary problems of calculus of variations with fixed right value and velocity of the state variable at the end instant T, for minimum of a cost functional. The integral cost functional describes discrepancy between measurements and real trajectories of the dynamical system. The functional is regularized by the small term. We use solutions of the auxiliary problems of calculus of variations to extract controls and apply them in the dynamical system. We consider the extracted controls as approximations of the solution to the control dynamic reconstruction problem.

In this paper, an illustrating example for flights at a prescribed altitude Letov (1969) is exposed.

2. STATEMENT

We consider controlled dynamical systems of the form

$$\frac{dx(t)}{dt} = f(t, x(t)) + G(t, x(t))u(t), \quad t \in [0, T], \tag{1}$$

where $x \in R^n$ are state variables, parameters $u \in R^n$ are restricted controls:

$$u \in U = \{u_i \in [a_i^-, a_i^+],\ a_i^- < a_i^+,\ i = 1, 2, \ldots, n\}. \tag{2}$$

It is assumed that information about inaccurate measurements of the state variable $x(t)$ of the realized (basic) solution $x^*(t)$ of equation (1)-(2) at discrete instants $t_i:\ t_0 = 0 < t_1 < \ldots < t_N = T$, and the relations are true

$$|x(t) - x^*(t)| \le \delta, \tag{3}$$

where $\delta \in (0, \delta_0]$ is the known estimation of measurement errors.

Let $t_i = t_0 + i\Delta t; \quad i = 1, \ldots, N, \quad \Delta t > 0.$

2.1 Assumptions

It is assumed that

A1. Coordinates $f_i(t, x),\ \ g_{i,j}(t, x),\ i, j = 1, \ldots, n$ of the vector-function $f(t, x)$ and the matrix-function $G(t, x)$ are defined and continuous in the strip $\Pi_T = [0, T] \times R^n$.

A2. The functions $f_i(t, x),\ \ g_{i,j}(t, x),\ \ i, j = 1, \ldots, n$ are continuously differentiable in domain $(0, T) \times R^n$ and their partial derivatives $\dfrac{\partial f_i(t, x)}{\partial t}$, $\dfrac{\partial f_i(t, x)}{\partial x_k}$, $\dfrac{\partial g_{ij}(t, x)}{\partial t}$, $\dfrac{\partial g_{ij}(t, x)}{\partial x_k}$, $k = 1, \ldots, n$ are extendable on any compact set $D \subset \Pi_T$.

A3. There are such a compact set $D_0 \subset \Pi_T$ and a number $r_0 > 0$ that

$$\{[0, T] \times \{x = 0\}\} \cap D_0 = \emptyset; \tag{4}$$

$$D_0 \supset \{(t, x) \in \Pi_T :\ \|x - x^*(t)\| \le r_0\}, \tag{5}$$

The symbol $\|\xi\|$ denotes Euclidean

A4. We assume that, at any current time $t \ge t_2$, for any $\delta \le \delta_0$, continuously differentiable functions $y^\delta()$: $[0, t] \to R^n$ can be defined, which are interpolations of measurements of the state variable $x^*(t)$, and

$$(t, x^*(t)) \in \Omega_\delta \subset D_0, \tag{6}$$

$$\Omega_\delta = \{(t, x) \in \Pi_T :\ |x - y^\delta(t)| \le 2\delta,\ t \in [t_0, T]\}. \tag{7}$$

A5. There exist a constant $K_1 > 0$ and a number $\delta_0 > 0$ such that the following conditions hold:

$$\|y^\delta(t)\| \le K_1, \quad \forall t \in [0, T]; \tag{8}$$

$$\|\frac{dy^\delta(t)}{dt}\| \le K_1, \quad \forall t \in [0, T] \setminus \Theta^\delta, \tag{9}$$

for any $\delta \in (0, \delta_0]$. The sets Θ^δ have the the measure $\beta^\delta = \beta(\Theta^\delta)$ and $\beta^\delta \to 0$, as $\delta \to 0$.

2.2 Dynamic reconstruction problem

The following dynamic reconstruction problem (DR) is under consideration.

DR *After receipt, at current instant $t \ge t_2$, information about interpolations of measurements $y^\delta(\cdot) : [0, t] \to R^n$, we have to reconstruct a continuous control $u^\delta(\cdot) : [0, t - \Delta t] \to U$ and the trajectory $x^\delta(\cdot) : [0, t - \Delta t] \to R^n$ of system (1)-(2), which has been generating by this control, such that*

$$(\tau, x^\delta(\tau)) \in D_0, \quad \forall \tau \in [0, t - \Delta t], \quad t \in [0, T], \tag{10}$$

and the relations

$$\|x^\delta(\cdot) - x^*(\cdot)\|_C = \max_{\tau \in [0, T - \Delta t]} \|x^\delta(\tau) - x^*(\tau)\| \to 0, \tag{11}$$

$$\|u^\delta(\cdot) - u^*(\cdot)\|_{L_2} = \int_0^{T - \Delta t} \|u^\delta(\tau) - u^*(\tau)\|^2 d\tau \to 0, \tag{12}$$

are hold, as $\delta \to 0$, $\Delta t \to 0$.

Here $\|\cdot\|_C$ is the norm in the space of continuous functions and $\|\cdot\|_{L_2}$ is the norm in the space L_2.

3. AUXILIARY CONSTRUCTIONS

To solve CRP (10)-(12), we introduce the following auxiliary calculus of variations problems (CV).

3.1 Calculus of variations problem

We consider an interval $[t_{i-2}, t_i],\ i = 2, \ldots, N$, a fixed parameter $\delta \in (0, \delta_0]$ and an interpolation $y^\delta(\tau),\ \tau \in [t_{i-2}, t_i]$, and introduce a cost functional $I = I_{t_{i-2}, x_0}(u(\cdot))$ of the form

$$I = \int_{t_{i-2}}^{t_i} \Big[-\frac{\|x(\tau) - y^\delta(\tau)\|^2}{2} + \frac{\alpha}{2}\|u(\tau)\|^2 \Big] d\tau, \tag{13}$$

where $\alpha > 0$ — is a small regularizing parameter, $(t_{i-2}, x_0) \in D_0 \subset \Pi_T$. A measurable function $u(\cdot)$ is the restriction to the segment $[t_{i-2}, t_i]$ of an admissible control in system (1) satisfied relations (2). The function $x(\tau),\ \tau \in [t_{i-2}, t_i]$ in (13) means the solution of system (1) which has been generating under the admissible control $u(\cdot)$ on the time interval $[t_{i-2}, t_i]$, and $x(t_{i-2}) = x_0$.

Now, for fixed $\delta \in (0, \delta_0]$, $\alpha > 0$, we solve the following calculus of variations problem (CV).

CV *We have to minimize the cost functional (13) over all solutions $x(\cdot)$ of system (1) satisfied the boundary conditions*

$$x(t_{i-2}) = x_0 = y^\delta(t_{i-2}), \quad \frac{dx(t_{i-2})}{dt} = \frac{dy^\delta(t_{i-2})}{dt}. \qquad (14)$$

3.2 Hamiltonian system

The hamiltonian of CV (13)-(14) has the form $H = H(t,x,s) =$

$$\min_{u\in R^n}\Big[s^\top f(t,x) + s^\top G(t,x)u - \frac{\|x(t)-y^\delta(t)\|^2}{2} + \frac{\alpha^2}{2}u^2\Big] =$$

$$\Big[s^\top f(t,x) + s^\top G(t,x)u^0 - \frac{\|x(t)-y^\delta(t)\|^2}{2} + \frac{\alpha^2}{2}{u^0}^2\Big] \qquad (15)$$

where $u^0 = u^0(t,x,s)$,

$$u^0 = -\frac{1}{\alpha^2}G^\top(t,x)s. \qquad (16)$$

The symbol $\top$ means transposition.

Necessary optimality conditions for $x(\cdot)$ in CV (13)-(14) have the form of the hamiltonian system

$$\frac{dx(t)}{dt} = H_s(t,x(t),s(t));$$
$$\frac{ds(t)}{dt} = -H_x(t,x(t),s(t)); \qquad (17)$$
$$t\in[t_{i-2},t_i], \quad i=2,\dots,N,$$

and the boundary conditions hold:

$$x(t_{i-2}) = y^\delta(t_{i-2}), \qquad (18)$$

$$s(t_{i-2}) = G^{-1}(t_{i-2},y^\delta(t_{i-2}))\frac{dy^\delta(t_{i-2})}{dt} -$$
$$-G^{-1}(t_{i-2},y^\delta(t_{i-2}))f(t_{i-2},y^\delta(t_{i-2})). \qquad (19)$$

Here

$$H_s(t,x,s) = s^\top f(t,x) + s^\top G(t,x)u^0;$$

$$H_x(t,x,s) = -(x(t)-y^\delta(t)) + s^\top D_x f(t,x) +$$
$$+ s^\top D_x G(t,x)u^0 + s^\top G(t,x)D_x u^0 + \alpha^2 {u^0}^\top D_x u^0;$$

$$D_x f(t,x) = \big(D_{x_1}f(t,x),\dots,D_{x_n}f(t,x)\big);$$

$$D_x G(t,x) = \big(D_{x_1}G(t,x),\dots,D_{x_n}G(t,x)\big);$$

$$D_x u^0 = \big(D_{x_1}u^0(t,x,s),\dots,D_{x_n}u^0(t,x,s)\big).$$

According to results of algebra Bellman (1960) the conditions

$$\xi^\top G^\top(t,x)G(t,x)\xi > 0, \qquad (20)$$

are true for all $(t,x)\in D_0,\ \xi\in R^n,\ \|\xi\|\neq 0$.

Note, that assumptions **A1**. – **A4**. imply existence of the unique solution $x(\cdot), s(\cdot)$, to problem (17)-(18).

4. SOLUTION OF DR

We get consequentially

$$\hat u^0 = \begin{cases} u^0(t,x,s) = -\dfrac{1}{\alpha^2}\hat G^\top(t,x)s, \quad u^0(t,x,s)\in V,\\[2mm] \hat u_i^0 = a_i^+, \quad u_i^0(t,x,s)\ge a_i^+,\ i\in\overline{1,m},\\ \hat u_j^0 = u_j^0(t,x,s),\ j\neq i;\\[2mm] \hat v_i^0 = a_i^-, \quad u_i^0(t,x,s)\le a_i^-,\ i\in\overline{1,m},\\ \hat u_j^0 = u_j^0(t,x,s),\ j\neq i. \end{cases} \qquad (21)$$

$$i\in\overline{1,n},\ \tau\in[t_{j-2},t_{j-1}],\ j\in\overline{2,N}$$

and consider $\hat u^0(\tau)$ as an approximation for control $u^*(\tau)$ reconstructed with a small delay Δt on the intervals $[t_{j-2},t_{j-1}],\ j\in\overline{2,N}$ for basic system (1)-(2).

Applying ideas and schemes of proof from papers Subbotina (2015); Subbotina et al. (2016); Subbotina and Krupennikov (2017) we can get the following assertion to justify the presented algorithm.

Theorem 1. If assumptions A.1–A.4 are true for problem DR, then the concordance between parameters $\delta, \alpha = \alpha(\delta)$, and Δt can be established in such a way, that controls $u^\delta(\tau) = u^{\delta,\alpha(\delta)}(\tau) = \hat u^0(\tau)$ of the form (21) satisfy the relations (10)–(12), i.e. they solve DR.

5. NUMERICAL EXAMPLE

We consider the model of flights at a prescribed altitude **?**. Dynamics of the ship is described by the system

$$\frac{dx_1}{dt} = x_2;$$
$$\frac{dx_2}{dt} = \frac{cu - Q(x_2)}{m(t)}; \qquad (22)$$

with the restrictions on controls

$$u\in U = \{0\le u\le\beta\}. \qquad (23)$$

Here x_1 is the great-circle arc starting from certain point, x_2 is the ship velocity along its trajectory, $m(t)$ is fuel mass, $m(t)\ge m_0>0$, control u is the fuel consumption limited from above , Q is the aerodynamic drag defining via the formula (see, **?**,

$$Q = c_x pS\frac{(x_2)^2}{2}. \qquad (24)$$

The symbol S denotes the wing area, p is the the air density, known as a function of altitude H at a constant temperature, the symbol c_x means the drag coefficient, which is given by the known function of $M = \frac{x_2}{a}$, where a is equal to the sound speed.

We watch the real landing process $(x_1^*(t), x_2^*(t))$ and get online inaccurate discrete state information $(y_1(t_j), y_2(t_j))$:

$$\|y_1(t_j) - x_1^*(t_j)\|\le\delta,$$

$t_0 = 0 < t_1 < t_2,\dots,< t_N = T,\ \delta>0$. We construct smooth continuous approximations $(y_1(t), y_2(t))$ of the measurements and apply the above presented method to get the approximation $u^0(t)$ of the reconstructing control $u^*(t)$ with a small delay.

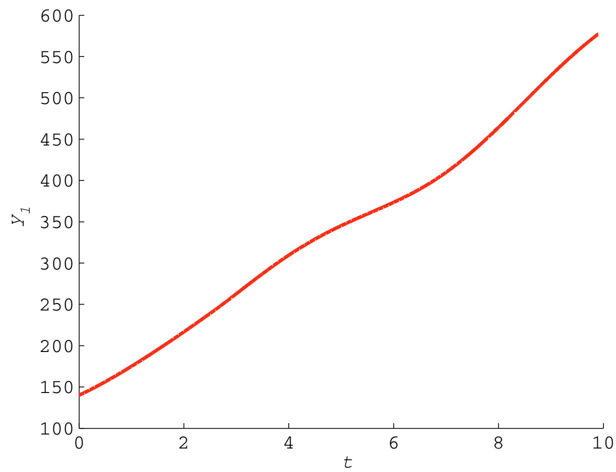

Fig. 1. State information $y_1(\cdot)$

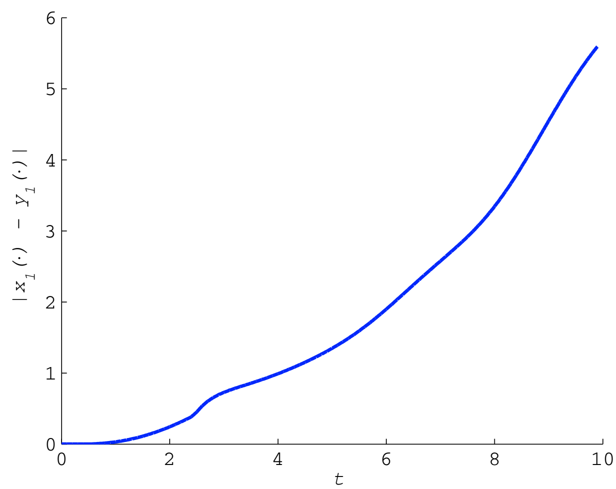

Fig. 2. Difference between state information $y_1(\cdot)$ and reconstructed trajectory $x_1(\cdot)$

There are results of numerical experiments in this section. We put $S = 10$, $p = 0.526(kg/m^3)$, $H = 8(km)$, $M = 0.4$, $c_x = 0.85$. We choose $\alpha = 0.5$, $\beta = 50$, $\delta = 0.001$

$$u^*(t) = \begin{cases} 10, 0 \leq t < 2.5, \\ 10 + 10\sin(t), 2.5 \leq t \leq 10. \end{cases} \quad (25)$$

Red lines on the pictures 1–5 are real control and trajectory, blue lines are the reconstructed control and the reconstructed trajectories.

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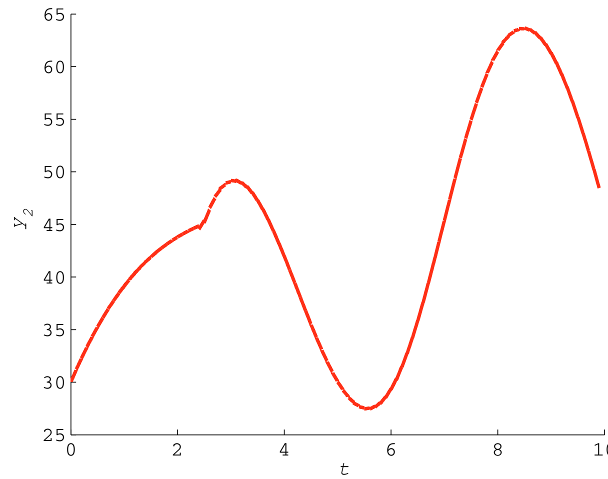

Fig. 3. State information $y_2(\cdot)$

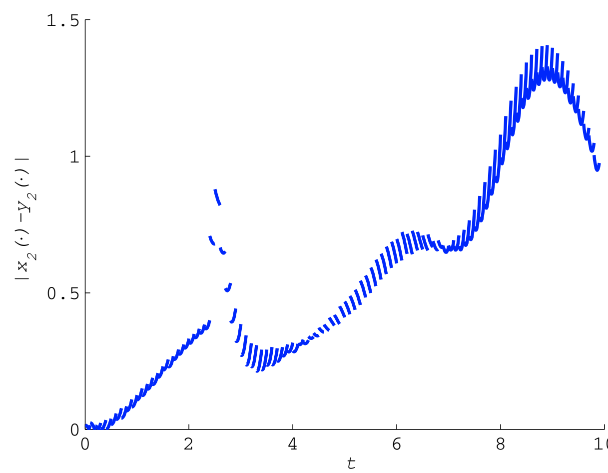

Fig. 4. Difference between state information $y_2(\cdot)$ and reconstructed trajectory $x_2(\cdot)$

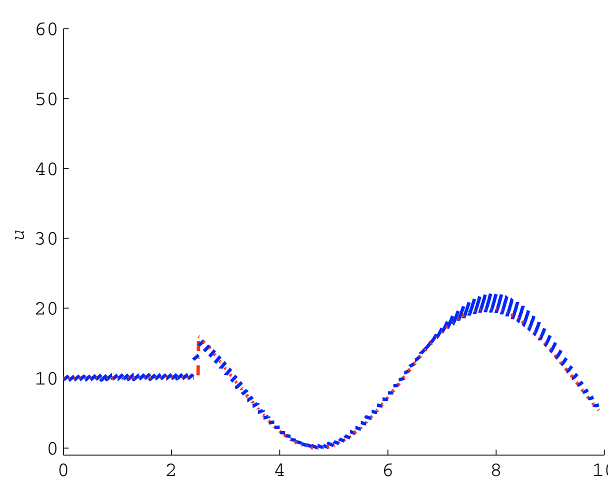

Fig. 5. Unknown control $u^*(\cdot)$ (red line) and reconstructed control $u^0(\cdot)$ (blue line)

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